SERRE SYMMETRY PERSISTS THROUGH THE FRÖLICHER SPECTRAL SEQUENCE

Abstract. Serre's duality theorem implies a symmetry between the Hodge numbers, $h^{p,q} = h^{n-p,n-q}$, on a compact complex n–manifold. Equivalently, the first page of the associated Frölicher spectral sequence satisfies $\dim E_1^{p,q} = \dim E_1^{n-p,n-q}$ for all p, q. Adapting an argument of Chern, Hirzebruch, and Serre [3] in an obvious way, in this short note we observe that this "Serre symmetry" $\dim E_k^{p,q} = \dim E_k^{n-p,n-q}$ holds on all subsequent pages of the spectral sequence as well. The argument will show that an analogous statement holds for the Frölicher spectral sequence of an almost complex structure on a nilpotent real Lie group as considered in [4].

1. Introduction

Associated to a complex manifold one has the so-called Frölicher or Hodge–to–de Rham spectral sequence [7] which relates the Dolbeault cohomology to the de Rham cohomology of the manifold. As a consequence, one obtains relations between the dimensions of the Dolbeault cohomology and topological invariants of the manifold in the compact case. Frölicher [7] observes the inequality $\sum_{p+q=k} h^{p,q} \geq b_k$, where $h^{p,q}$ denotes the dimension of the Dolbeault cohomology group $H^{p,q}_{\bar\partial}$, and b_k is the k^{th} Betti number, along with the equality $\sum_{p,q}(-1)^{p+q}h^{p,q} = \sum_i(-1)^i b_i$, with the right-hand term being the topological Euler characteristic. The inequality becomes an equality for all k if and only if the spectral sequence degenerates on its first page, which happens e.g. if the complex structure is Kähler or more generally, Moišezon (see e.g. [6]). In the past few decades there has been interest in studying examples of compact complex manifolds for which the Frölicher spectral sequence degenerates only on the second page or later (see e.g. [2],[5]), with complex nilmanifolds being most amenable to this study. In this short note, we will observe through an adaptation of the arguments in [3] that the symmetry $\dim E_1^{p,q} = \dim E_1^{n-p,n-q}$ we have on the first page of the spectral sequence, due to Serre duality on a compact complex n–manifold, continues to hold on all subsequent pages, thereby significantly reducing the amount of computation required to obtain these dimensions.

2. Serre symmetry

Definition 2.1. (cf. [3]) An n–dimensional *bigraded Poincaré ring* is a bigraded ring A with the following properties:

(1) Each $A^{p,q}$ is a finite-dimensional complex vector space, and there exists an n such that $A^{p,q} = \{0\}$ if $p > n$ or $q > n$ and $A^{n,n}$ is one-dimensional.

(2) Denote $A^k = \bigoplus_{p+q=k} A^{p,q}$. If $x \in A^k$ and $y \in A^l$ then $xy = (-1)^{kl} yx$.

(3) Fix a nonzero element $\xi \in A^{n,n}$. Define a bilinear pairing $A^{p,q} \otimes A^{n-p,n-q} \xrightarrow{\langle -,- \rangle} \mathbb{C}$ by $\langle x, y \rangle \xi = xy$. We require the linear map $i_{n-p,n-q}$ from $A^{n-p,n-q}$ to $A^{p,q*}$ (the complex-linear dual of $A^{p,q}$) which sends $y \in A^{n-p,n-q}$ to the functional $\langle -, y \rangle$ to be an isomorphism for all p, q. (Note that this property does not depend on the choice of nonzero ξ.)

Example 2.2. For X a compact complex manifold of complex dimension n, the Dolbeault cohomology $H^{*,*}_{\bar\partial}$ is a bigraded Poincaré ring. Indeed, the space of (p,q) forms for p, q not between 0 and n is trivial; each $H^{p,q}_{\bar\partial}$ is finite-dimensional by elliptic theory; $H^{n,n}_{\bar\partial}$ is one-dimensional, spanned by a volume form for the manifold. Therefore property (1) of the definition is satisfied. Property (2) follows from the same property holding on the algebra of all complex-valued differential forms on X and the fact that $\bar\partial$ is a derivation with respect to the product of forms. As for property (3), let $\xi = [\Omega]_{\bar\partial}$ where Ω is some fixed (n,n) volume form on X. Since $\bar\partial\Omega = 0$ and it is not $\bar\partial$-exact (since otherwise it would be d-exact for degree reasons, and this cannot be by Stokes' theorem), ξ spans $H^{n,n}_{\bar\partial}$. Now, we will show that $i_{n-p,n-q}$ is an isomorphism; since by Serre duality $\dim H^{p,q}_{\bar\partial} = \dim H^{n-p,n-q}_{\bar\partial}$, it will suffice to show injectivity. Take a non-zero $\alpha \in H^{n-p,n-q}_{\bar\partial}$ and take the $\bar\partial$-harmonic representative y of this class. Denote by $\overline{*y}$ the conjugate of its Hodge dual, i.e. the (p,q)-form such that $y \wedge \overline{*y} = \langle y, y \rangle_{L^2} \Omega$. Since $*\partial * y = -\bar\partial^* y = 0$, we have $\partial * y = 0$ and hence $\bar\partial \overline{*y} = 0$. Now $y \wedge \overline{*y} = ||y||_{L^2}\Omega$, and $||y||_{L^2} \neq 0$, and thus passing to cohomology we obtain that $[y]_{\bar\partial} \wedge [\overline{*y}]_{\bar\partial}$ is a non-zero multiple of ξ. We conclude $\langle [\overline{*y}]_{\bar\partial}, [y]_{\bar\partial} \rangle \neq 0$, i.e. $i_{n-p,n-q}(\alpha) \neq 0$.

Definition 2.3. (cf. [3]) A *differential* on an n–dimensional bigraded Poincaré ring A is a linear map $A \xrightarrow{d} A$ satisfying:

(α) $d(A^{p,q}) \subset A^{p+p',q+q'}$, i.e. d is of bidegree (p', q') for some $p', q' \in \mathbb{Z}$. We further require $p' + q' = 1$ (but not necessarily $p', q' \geq 0$).

(β) $d^2 = 0$.

(γ) $d(xy) = (dx)y + (-1)^r x(dy)$ for $x \in A^r$, $r \in \mathbb{Z}$.

(δ) $d(A^{2n-1}) = \{0\}$.

Example 2.4. For X a compact complex manifold, the induced differential ∂ on $A = H_{\bar\partial}(X)$ (given by $\partial[\alpha]_{\bar\partial} = [\partial\alpha]_{\bar\partial}$) is a differential of bidegree $(1,0)$ on the bigraded Poincaré ring $H_{\bar\partial}(X)$ in the above sense. Indeed, property (α) is satisfied, and properties (β) and (γ) are satisfied by ∂ on the level of forms (note that the derivation ∂ on forms induces a derivation on $H_{\bar\partial}(X)$). As for property (δ), we note that $A^{2n-1} = A^{n-1,n} \oplus A^{n,n-1}$. The differential vanishes on $A^{n,n-1}$ for degree reasons. Now take $[\alpha]_{\bar\partial} \in A^{n-1,n}$ and suppose $\partial[\alpha]_{\bar\partial} = [\partial\alpha]_{\bar\partial}$ is non-zero. Since $A^{n,n}$ is one-dimensional, this means there is a non-zero constant c such that $\partial\alpha - c\Omega = \bar\partial\beta$ for some $(n, n-1)$-form β. Rearranging, we get $\Omega = \partial(\frac{\alpha}{c}) - \bar\partial(\frac{\beta}{c})$. For degree reasons, this is the same as $\Omega = d(\frac{\alpha-\beta}{c})$, which by Stokes' theorem would imply $X = \emptyset$. (Alternatively, the vanishing of this differential follows from the convergence of the Frölicher spectral sequence to the complexified de Rham cohomology and $H^{2n}_{dR}(X;\mathbb{C}) \cong \mathbb{C}$.)

Proposition 2.5. *(cf. [3]) The cohomology ring $HA = \ker d/\operatorname{im} d$ of an n-dimensional bigraded Poincaré ring A with differential d is an n-dimensional Poincaré ring.*

Proof. Since d has a well-defined bidegree we obtain a decomposition $HA = \bigoplus_{p,q} {HA'}^{p,q}$, where $HA^{p,q} = (\ker d \cap A^{p,q})/\operatorname{im} d$. Note that property (δ) implies that HA inherits properties (1) and (2). If ξ is the (n,n)-bidegree element used to define $\langle -,-\rangle$ on A, then we can define a bilinear pairing $HA^{p,q} \otimes HA^{n-p,n-q} \xrightarrow{\langle -,-\rangle} \mathbb{C}$ on the cohomology ring by $\langle [x],[y]\rangle[\xi] = [x][y]$. Note that $[\xi]$ is non-zero due to property (δ). Now define a linear map $HA^{n-p,n-q} \xrightarrow{i'_{n-p,n-q}} (HA^{p,q})^*$ by sending $[y]$ to the functional $\langle -,[y]\rangle$.

Now we show that HA has property (3), i.e. that $i'_{n-p,n-q}$ is an isomorphism for all p,q. Denote the bidegree of d by (p',q'); recall $p'+q' = 1$. For $x \in A^{p,q}$ and $y \in A^{n-p-p',n-q-q'}$, by properties (γ) and (δ) of the differential, we have $0 = d(xy) = (dx)y + (-1)^{p+q}x(dy)$. From the definition of $\langle -,-\rangle$, this implies $\langle dx, y\rangle = (-1)^{p+q-1}\langle x, dy\rangle$. Consider the following diagram (extending to left and right):

$$\begin{tikzcd}
{} \arrow[r] & A^{n-p-p',n-q-q'} \arrow[r, "d"] \arrow[d, "i_{n-p-p',n-q-q'}"] & A^{n-p,n-q} \arrow[r, "d"] \arrow[d, "i_{n-p,n-q}"] & A^{n-p+p',n-q+q'} \arrow[r] \arrow[d, "i_{n-p+p',n-q+q'}"] & {} \\
{} \arrow[r] & A^{p+p',q+q'*} \arrow[r, "(-1)^{p+q-1}d^*"] & A^{p,q*} \arrow[r, "(-1)^{p+q}d^*"] & A^{p-p',q-q'*} \arrow[r] & {}
\end{tikzcd}$$

Here d^* denotes the dual morphism to d; we have $d^*(\langle -,y\rangle) = \langle d-,y\rangle$. Note that the calculation preceding the diagram implies that i is a chain map for all p,q. Indeed, let us check commutativity for the left square: for

$y \in A^{n-p-p',n-q-q'}$ we have $i_{n-p,n-q}(dy) = \langle -, dy\rangle$, while

$$(-1)^{p+q-1}d^*(i_{n-p-p',n-q-q'}(y)) = (-1)^{p+q-1}d^*(\langle -, y\rangle) = (-1)^{p+q-1}\langle d-, y\rangle.$$

Since $i_{n-p,n-q}$ is an isomorphism for all p, q by property (3), it follows that the chain map i is a quasi-isomorphism. We would now like to relate this induced map to the map $i'_{n-p,n-q}$ we wish to establish bijectivity for.

First of all, for $y \in \ker d \cap A^{n-p,n-q}$, we have $i_{n-p,n-q}(y) = \langle -, y\rangle$, and so the induced map $i^*_{n-p,n-q}$ sends $[y]$ to the d^*-cohomology class $[\langle -, y\rangle]$. Now we note that the target space $\ker d^* \cap (A^{p,q})^*/\operatorname{im} d^*$ is not quite $(HA^{p,q})^* = (\ker d \cap A^{p,q}/\operatorname{im} d)^*$. Regardless, by the universal coefficient theorem, the map Ψ from the latter to the former given by $\Psi([\alpha])([V]) = \alpha(V)$ is an isomorphism (since we are over a field). Given $[x] \in HA^{p,q}$, note that $\Psi \circ i^*_{n-p,n-q}([y])([x])$ equals $\Psi([\langle -, y\rangle])([x]) = \langle x, y\rangle$. However, as $\langle x, y\rangle\xi = xy$, passing to cohomology we see that $\langle x, y\rangle = \langle [x], [y]\rangle$. (For this equality we are using that the pairing on the cohomology ring is defined using $[\xi]$.) Therefore $\Psi \circ i^*_{n-p,n-q} = i'_{n-p,n-q}$. Since both maps on the left hand side are isomorphisms, we have the desired statement. □

Corollary 2.6. *The E_2 page of the Frölicher spectral sequence for a compact complex manifold satisfies Serre duality. (The E_2 page here denotes the ∂-cohomology of the $\bar\partial$-cohomology.)*

Recall that for a complex manifold X, denoting its algebra of complex-valued smooth forms by $A^{\bullet,\bullet}(X)$, the Frölicher spectral sequence is associated to the filtration $F^pA^{\bullet,\bullet}(X) = \bigoplus_{i\geq p,j} A^{i,j}(X)$. The complexified de Rham differential d preserves this filtration, $d(F^p) \subset F^p$, and if $\alpha \in F^p$, $\beta \in F^q$, then $\alpha \wedge \beta \in F^{p+q}$. Now by [8, Theorem 2.14], it follows that the differential on the E_k page of the Frölicher spectral sequence, for any $k \geq 1$, satisfies property (γ). We know that it also satisfies properties (α) with bidegree $(k, 1-k)$ and (β). Property (δ) is trivially satisfied for degree reasons (or by the alternative argument as given in Example 4). In particular, (E_2, d_2) is a bigraded Poincaré ring with differential, and so inductively we have the following:

Corollary 2.7. *For every $k \geq 1$, the kth page of the Frölicher spectral sequence of a compact complex n–manifold satisfies Serre symmetry, i.e.* $\dim E_k^{p,q} = \dim E_k^{n-p,n-q}$ *for all p, q.*

In [4], the authors define a Dolbeault cohomology for almost complex manifolds, which in the case of an integrable complex structure coincides with the usual Dolbeault cohomology. Namely, on an almost complex manifold (M, J) the exterior differential d splits into four bigraded pieces, $d = \bar\mu + \bar\partial + \partial + \mu$, where $\bar\mu$ is of bidegree $(-1, 2)$. From $d^2 = 0$ it follows that

$\mu^2 = 0$, $\bar\partial\bar\mu + \bar\mu\bar\partial = 0$, and $\bar\mu\partial + \partial\bar\mu + \bar\partial^2 = 0$; hence $H_{\bar\mu}(M,J) = \ker\bar\mu/\operatorname{im}\bar\mu$ is well defined and $\bar\partial$ descends to a map on $H_{\bar\mu}(M,J)$ which satisfies $\bar\partial^2 = 0$ (with the $\bar\mu$-chain homotopy between $\bar\partial^2$ and 0 being provided by ∂). The Dolbeault cohomology of (M,J) is then defined to be $H_{\bar\partial}H_{\bar\mu}(M,J)$. An analogous notion is defined for real Lie algebras with an almost complex structure, and in this case the Dolbeault cohomology groups are known to satisfy a Serre duality if, e.g. the Lie algebra is nilpotent [4, Corollary 5.5]. The above argument for compact complex manifolds carries through in this case as well, and we obtain the following observation on the Frölicher spectral sequence associated to the shifted Hodge filtration as considered in [4, Section 3].

Corollary 2.8. *For every $k \geq 1$, the kth page of the Frölicher spectral sequence of a real $2n$–dimensional nilpotent Lie group with left-invariant almost complex structure satisfies Serre symmetry, i.e.* $\dim E_k^{p,q} = \dim E_k^{n-p,n-q}$ *for all p, q.*

Remark 2.9. The result of Corollary 2.7 has been known to Stelzig (2018) from his study of double complexes [10] and Popovici (2017) via harmonic theory (see Popovici–Stelzig–Ugarte [9]) . We note that this result shortens some of the calculations done in [1]. It is as of yet unknown whether Serre symmetry holds in general for the Dolbeault cohomology of a (not necessarily integrable) almost complex structure on a closed manifold as studied in [4].

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